MATH 245 F24, Exam 2 Solutions

1. Carefully define the following terms: Proof by Shifted Induction, recurrence.
   Let $s \in \mathbb{Z}$. To prove "$\forall x \in \mathbb{Z}$ with $x \geq s$, $P(x)$" by shifted induction, we must (a) prove $P(s)$; and (b) prove $\forall x \in \mathbb{Z}$ with $x \geq s$, $P(x) \to P(x+1)$. A recurrence is a sequence in which all but finitely many terms are defined out of the previous terms.

2. Carefully state The Master Theorem.
   Let $a \in \mathbb{N}$, $k \in \mathbb{R}$, and $b \in \mathbb{R}$ with $b > 1$ (three constants). Let c_n be a sequence with $c_n = \Theta(n^k)$. Set $d = \log_b a$. Consider now the sequence $T_n = aT_{n/b} + c_n$. The Master Theorem now has three cases:
   If $k < d$, then $T_n = \Theta(n^d)$. If $k = d$, then $T_n = \Theta(n^d \log n)$. If $k > d$ then $T_n = \Theta(n^k)$.

3. Let $a_n = n^3 + 4n + \sin(n)$. Prove or disprove that $a_n = \Theta(n^3)$.
   The statement is true, and its proof has two halves.

   $a_n = O(n^3)$: Choose $M = 6$ and $n_0 = 1$. Let $n \geq n_0$. Now, $n^3 \geq n$ so $4n^3 \geq 4n$. Also, $n^3 \geq n \geq 1$. Now, $|a_n| \leq |n^3| + |4n| + |\sin n| \leq n^3 + 4n + 1 \leq n^3 + 4n^3 + n^3 = 6n^3 = 6|b_n|$.

   $a_n = \Omega(n^3)$: Choose $M = 1$ and $n_0 = 1$. Let $n \geq n_0$. Now, $\sin n \geq -1$, so $n + \sin n \geq n - 1 \geq 0$, and hence $n^3 + 3n + (n + \sin n) \geq 0$. Hence, $1|a_n| = |n^3 + 4n + \sin n| = n^3 + 4n + \sin n = n^3 + 3n + (n + \sin n) \geq n^3 + 0 + 0 = |b_n|$.

   COMMON ALGEBRA ERRORS: While it is true that $|a + b| \leq |a| + |b|$ (this is called the triangle inequality), it is NOT true that $|a + b| = |a| + |b|$. For example, take $a = 3, b = -3$; we have $|a + b| = 0$ while $|a| + |b| = 6$. It is also NOT true that $|\sin n| = \sin n$. For example, take $n = -\pi/2$; we have $|\sin n| = 1$ while $\sin n = -1$. Some of you got these errors out of your system when you were solving exercise 7.15, but some of you did not.

4. Let $n \in \mathbb{Z}$. Prove that $\frac{n^2 - n - 2}{2} \in \mathbb{Z}$.
   Note that $n^2 - n - 2 = (n - 2)(n + 1)$. We apply the Division Algorithm Theorem to $n, 2$, getting $q, r \in \mathbb{Z}$ with $n = 2q + r$ and $0 \leq r < 2$.
   (or we can cite Cor. 1.8 to know that n is even or odd, then use the definitions of even, odd).

   If $n = 2q + 0$, then $\frac{n^2-n-2}{2} = \frac{(2q-2)(n+1)}{2} = (q - 1)(n + 1) \in \mathbb{Z}$.
   If instead $n = 2q + 1$, then $\frac{n^2-n-2}{2} = \frac{(n-2)(2q+1+1)}{2} = (n - 2)(q + 1) \in \mathbb{Z}$.
   In both cases, $\frac{n^2-n-2}{2} \in \mathbb{Z}$.

5. Prove that for all $n \in \mathbb{N}$, we must have $4^n \geq 3^n + 1$.
   Proof by (vanilla) induction. Base case, $n = 1$, $4^n = 4 \geq 3 + 1 = 3^n + 1$.
   Inductive case, let $n \in \mathbb{N}$ and assume that $4^n \geq 3^n + 1$. Multiply both sides by 4 to get $4^{n+1} = 4 \cdot 4^n \geq 4 \cdot 3^n + 4 \geq 4 \cdot 3^n + 1 \geq 3 \cdot 3^n + 1 = 3^{n+1} + 1$. Hence $4^{n+1} \geq 3^{n+1} + 1$.

   ALTERNATE ALGEBRA: Multiply both sides by 3 to get $3 \cdot 4^n \geq 3 \cdot 3^n + 3$. Now, $4^{n+1} = 4 \cdot 4^n > 3 \cdot 4^n$ and also $3 \cdot 3^n + 3 > 3 \cdot 3^n + 1 = 3^{n+1} + 1$. Combine everything to get $4^{n+1} \geq 3^{n+1} + 1$.

6. Prove that the Fibonacci numbers satisfy, for all $n \in \mathbb{N}_0$, that $F_nF_{n+1} = \sum_{i=0}^{n} F_i^2$.

Proof by (shifted) induction. Base case $n = 0$: $F_0F_1 = 0{\cdot}1 = 0$, while $\sum_{i=0}^{0} F_i^2 = F_0^2 = 0^2 = 0$. Inductive case, let $n \in \mathbb{N}_0$ and assume that $F_nF_{n+1} = \sum_{i=0}^{n} F_i^2$. Add F_{n+1}^2 to both sides, getting $F_{n+1}^2 + F_nF_{n+1} = \sum_{i=0}^{n+1} F_i^2$. Now, $F_{n+1}^2 + F_nF_{n+1} = F_{n+1}(F_{n+1} + F_n) = F_{n+1}F_{n+2}$. Combining, we get $F_{n+1}F_{n+2} = \sum_{i=0}^{n+1} F_i^2$.

7. Solve the recurrence with initial conditions $a_0 = 3, a_1 = -1$, and relation $a_n = a_{n-1} + 6a_{n-2}$ (for $n \geq 2$).

The characteristic polynomial is $x^2 - x - 6 = (x - 3)(x + 2)$, whose roots are $3, -2$. Hence the general solution is $a_n = A3^n + B(-2)^n$. We now apply the initial conditions $3 = a_0 = A3^0 + B(-2)^0 = A + B$ and $-1 = a_1 = A3^1 + B(-2)^1 = 3A - 2B$. We now solve the system $\{3 = A + B, -1 = 3A - 2B\}$ to get $A = 1, B = 2$. Hence, the specific solution is $a_n = 3^n + 2(-2)^n$.

8. Prove or disprove: $\forall x \in \mathbb{Z}, !y \in \mathbb{Z},\ x = y^2 + y$.

The statement is false; a counterexample consists of an x and two different y's. Many choices are possible: $(x = 0, y_1 = 0, y_2 = -1), (x = 2, y_1 = 1, y_2 = -2), (x = 6, y_1 = 2, y_3 = -3)$, etc. In all cases, a complete solution must show that $x = y_1^2 + y_1$ and $x = y_2^2 + y_2$.

One full solution: Take $x = 2, y_1 = 1, y_2 = -2$. We have $y_1^2 + y_1 = 1^2 + 1 = 2 = x$ and also $y_2^2 + y_2 = (-2)^2 + (-2) = 4 - 2 = 2 = x$. Since $y_1 \neq y_2$, the statement is false.

9. Let $x, y \in \mathbb{R}$. Without using any theorems from Chapter 5, prove that: if $x \leq y$ then $\lceil x \rceil \leq \lceil y \rceil$.

Direct proof: suppose that $x \leq y$. We apply the definition of ceiling twice: $\lceil x \rceil - 1 < x \leq \lceil x \rceil$ and $\lceil y \rceil - 1 < y \leq \lceil y \rceil$. We combine the first inequality with x and the second inequality with y, to get the chain $\lceil x \rceil - 1 < x \leq y \leq \lceil y \rceil$, and hence $\lceil x \rceil - 1 < \lceil y \rceil$. Since $\lceil x \rceil - 1$ and $\lceil y \rceil$ are both integers, we can apply Theorem 1.12(a) from the book (you don't need to know the number, just that it's a theorem and it's not in chapter 5), getting $\lceil x \rceil - 1 \leq \lceil y \rceil - 1$. Adding 1 to both sides we get the desired $\lceil x \rceil \leq \lceil y \rceil$.

10. Let $a, b \in \mathbb{Z}$ with $b \geq 1$. Use maximum element induction to prove that there are $q, r \in \mathbb{Z}$ satisfying $a = 2bq + r$ and $-b < r \leq b$.

We set $S = \{m \in \mathbb{Z} : m < \frac{a+b}{2b}\}$. The real number $\frac{a+b}{2b}$ is an upper bound for S, and S is nonempty because it is a halfline. By maximum element induction, there is some maximum to S, i.e. some integer q such that $q < \frac{a+b}{2b}$ but $q + 1 \geq \frac{a+b}{2b}$. Set $r = a - 2bq$. We have $2bq < a + b$ and hence $-b < a - 2bq = r$. We also have $2bq + 2b \geq a + b$ so $b \geq a - 2bq = r$. Combining, we get $-b < r \leq b$, as desired. Since $r = a - 2bq$, we also have $a = 2bq + r$.